EFFECT OF THERMOMECHANICAL CONNECTEDNESS ON THE DYNAMIC BEHAVIOR OF VISCOELASTIC BODIES

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A special characteristic of the behavior of viscoelastic materials is their capacity for dissipation of a considerable part of the mechanical energy supplied and a considerable dependence of their physicomechanical properties on the temperature. The effect of thermomechanical connectedness comes out most clearly in the process of long-term periodic deformations. The present article, on the basis of the determining equations of the thermomechanical theory of viscoelasticity [1], gives an approximate statement of the forced vibrations of nonlinear viscoelastic bodies. With the framework of this statement, an investigation is made of some thermomechanical effects due to dissipation, as well as to the dependence of the properties of the material on the temperature and the amplitude of the deformation.

1. In accordance with [2, 3], the determining equations of a broad class of media are connected with the assignment of certain functions (functionals) of state, e.g., the Helmholtz specific free energy ψ. From the point of view of the possibilities of adequate modeling of the medium and of the relative simplicity of the experimental program, theories based on a single-integral representation of the assigned functions have a definite advantage. For so-called generalized thermorheologically simple materials [1], the principal determining assumption has the form

$$\psi = \psi^{\infty}(E, T) + \int_{-\infty}^{t} N(E_d, T_d, E, T, \zeta)\, a[\Lambda(\tau)]\, d\tau, \tag{1.1}$$

here

$$N(O, 0, E, T, \zeta) = 0.$$

Here $E = (1/2)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$ is an infinitesimal deformation; T is the absolute temperature; ψ^{∞} is the equilibrium energy; $(\Lambda(\tau)) = (T(\tau), E(\tau), \dot{E}(\tau), \ldots, \overset{(n)}{E}(\tau))$ is a set of arguments; $E_d = E(\tau) - E$ are difference histories; ζ is the difference reduced time, determined by the relationship

$$(\xi, \xi') = \int_{0}^{(t,\tau)} a[\Lambda(z)]\, dz, \quad \zeta = \xi - \xi' = \int_{\tau}^{t} a[\Lambda(z)]\, dz,$$

($a > 0$ is a scalar function).

The application of standard thermodynamic formalism to (1.1) leads to equations for the stress S, the specific entropy η, and the internal dissipation σ

$$\begin{aligned} \frac{1}{\rho} S &= \psi^{\infty}_{,E} + \int_{-\infty}^{t} (N_{,E} - N_{,E_d})(\Gamma)\, a[\Lambda(\tau)]\, d\tau, \\ \eta &= -\psi^{\infty}_{,T} - \int_{-\infty}^{t} (N_{,T} - N_{,T_d})(\Gamma)\, a[\Lambda(\tau)]\, d\tau, \\ T\sigma &= -a[\Lambda(t)] \int_{-\infty}^{t} N_{,\zeta}(\Gamma)\, a[\Lambda(\tau)]\, d\tau, \quad \sigma \geqslant 0, \end{aligned} \tag{1.2}$$

where ρ is the density; $(\ )_{,X}$ denotes a partial derivative with respect to X.

We take the equations of motion and energy in the form

$$\begin{aligned} \operatorname{div} S + \rho \mathbf{b} &= \rho \ddot{\mathbf{u}}, \\ -\operatorname{div} \mathbf{h} + \rho r + \rho T \sigma &= \rho T \dot{\eta}, \end{aligned} \tag{1.3}$$

where $\mathbf{b}$ is the vector of the mass forces; $\mathbf{h}$ is the heat flux; r is the specific heat source. Relationships (1.2), (1.3), supplemented by the equations for E and h, form a closed system of equations of nonlinear viscoelasticity.

2. We assume that, up to the moment $\tau = 0$, the body was in an isothermal natural state, while, with $\tau > 0$, it was subjected to harmonic perturbation with the frequency ω. To construct an approximate theory of the thermomechanical behavior of bodies we assume that the temperature changes only slightly after a cycle of vibrations, that the duration of the transitional process of not fully established vibrations is small, and that the amplitudes of the modes of fully established vibrations with frequencies differing from ω are small. These hypotheses make it possible to approximately replace the

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values T, η, Tσ, h, and $\boldsymbol{a}$ by values averaged over a period

$$(\overline{T}, \overline{\eta}, \overline{T\sigma}, \overline{\mathbf{h}}, \overline{r}, \overline{a}) = \frac{\omega}{2\pi} \int\limits_{\tau}^{\tau+2\pi/\omega} (T, \eta, T\sigma, \mathbf{h}, r, a)\, dz, \tag{2.1}$$

and to represent the variables u, E, S, and b in form of the sum of averaged and oscillating components

$$\mathbf{u}(\mathbf{x}, \tau) = \overline{\mathbf{u}}(\mathbf{x}, \tau) + \operatorname{Re}\,[\tilde{\mathbf{u}}(\mathbf{x}, \tau)e^{i\omega\tau}], \tag{2.2}$$

where $\tilde{u} = u_1 + iu_2, \ldots$ are complex amplitudes. After substitution of formulas (2.1), (2.2) into the system (1.2). (1.3), we use the procedure of averaging for an approximate separation of the system into equations for the complex amplitudes and the averaged variables. If we are interested in the self-heating temperature, the variables $\overline{u}$, $\overline{E}$, and $\overline{S}$ can be discarded; then, the starting system of integrodifferential equations is reduced to a system of differential equations, including the equations of motion and energy

$$\operatorname{div} \tilde{S} + \rho\tilde{\mathbf{b}} + \rho\omega^2\tilde{\mathbf{u}} = 0; \tag{2.3}$$

$$-\operatorname{div} \overline{\mathbf{h}} + \rho\overline{r} + D = \rho\overline{T}\dot{\overline{\eta}}, \tag{2.4}$$

determining the equations

$$\tilde{S} = \tilde{G}[\tilde{E}]; \tag{2.5}$$

$$\overline{\mathbf{h}} = -k(\overline{T})\nabla\overline{T}, \tag{2.6}$$

to which must be added a relationship for $\tilde{E}$, as well as appropriate boundary and initial (for $\overline{T}(\tau)$) conditions, where k is the tensor of the thermal conductivity; ∇ is a nabla operator; $\tilde{G} = G' + iG''$ is a complex modulus; D is an averaged dissipative function, determined by the relationships

$$\frac{\rho\omega}{\pi} \int\limits_{0}^{2\pi/\omega} S(\tau) \begin{bmatrix} \cos\omega\tau \\ -\sin\omega\tau \end{bmatrix} d\tau = \begin{bmatrix} G'[E_1] - G''[E_2] \\ G'[E_2] + G''[E_1] \end{bmatrix},$$

$$D = \rho\overline{T\sigma} = (\omega/2)\{G''[E_1]\cdot E_1 + G''[E_2]\cdot E_2\}, \ \tilde{E} = E_1 + iE_2,$$

where

$$G'^{('')} = G'^{('')}(\overline{T}, \ \omega/\overline{a}, E_1, E_2); \ G[A] = G_{ijkl}A_{kl}; \ G[A]\cdot B = G_{ijkl}A_{kl}B_{ij}.$$

In distinction from the theory of thermorheologically simple materials, the complex modulus in (2.5) depends on the amplitude of the deformation; this dependence can either be explicit, or through the function a.

Using representations of the type of (1.1) for the Gibbs specific free energy $\rho g = SE - \rho\psi$, we can construct the relationships of the thermomechanical theory in terms of the creep.

3. Using the theory set forth above and its simplified variants, it is possible to investigate a broad range of thermomechanical phenomena, of which the most interesting is the phenomenon of thermal instability with cyclic loading. By thermal instability there is understood a sharp unbounded rise in the temperature of the body with the time in the case of an increase in the critical value λ_* of some parameter of the loading λ. In a quasistatic statement, this phenomenon was studied in [4].

Taking account of the forces of inertia and the amplitude dependence of the complex modulus can lead to qualitatively new thermomechanical effects. Let us first consider the problem of the dynamic behavior of a beam made of a thermorheologically simple material with properties not depending on the amplitude. It is postulated that the stresses $s_0 \cos \omega t$ are given at the ends of the beam.

The lateral surface and the end of the beam x = 0 are heat insulated, while the end $x = l$ is maintained at the temperature T_0 = const. Using the approximation for the complex Young modulus [5]

$$E^{*-1} = (c_1 - ic_2)\omega^\beta(T - T_1)^\gamma,$$

after the introduction of dimensionless quantities, we obtain the equations of motion (2.3) in terms of the stresses

$$p_1'' + (1+\theta)^\gamma (b_1p_1 + b_2p_2) = 0, \quad p_2'' + (1+\theta)^\gamma (b_1p_2 - b_2p_1) = 0, \tag{3.1}$$

the steady-state equation of energy (2.4)

$$\theta'' + b_2(1+\theta)^\gamma (p_1^2 + p_2^2) = 0 \tag{3.2}$$

and the boundary conditions

$$\begin{aligned} p_1 &= p_0, \ p_2 = 0, \ \theta' = 0 \text{ with } \xi = 0, \\ p_1 &= p_0, \ p_2 = 0, \ \theta = 0 \text{ with } \xi = 1, \end{aligned} \tag{3.3}$$

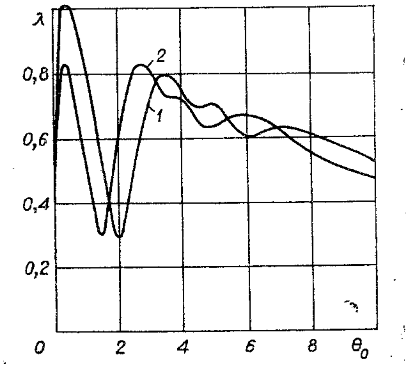

Fig. 1

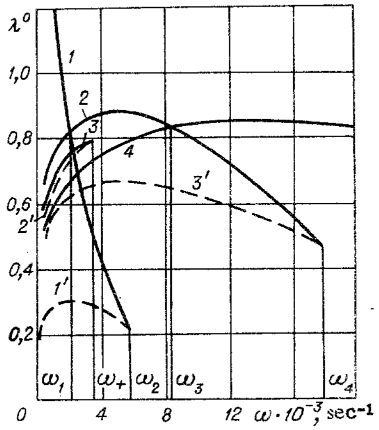

Fig. 2

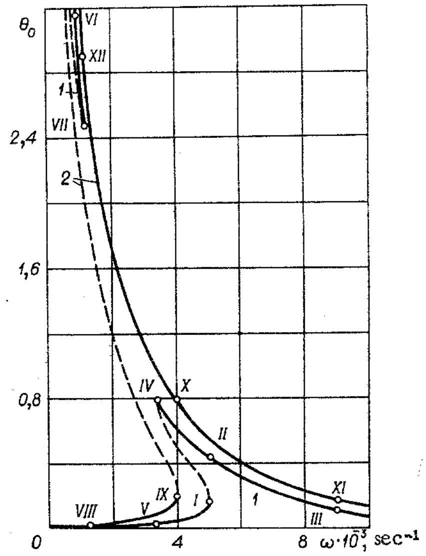

Fig. 3

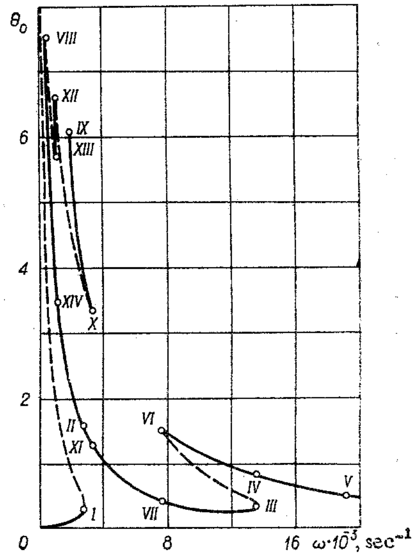

Fig. 4

where $p_{0,1,2} = \alpha s_{0,1,2}$; $\tilde{s} = s_1 + is_2$ is the amplitude of the axial stress; $\alpha = (2k\rho\omega T_2)^{-1/2}$; $\theta = (\bar{T} - T_0)/T_2$; $T_2 = T_0 - T_1$; $\xi = x/l$; $b_{1,2} = c_{1,2}\rho l^2\omega^{2+\beta}T_2^{\gamma}$; k is the thermal conductivity; $c_{1,2}$, β, γ, T_1 are constants of the material. For solution of the problem we use a difference approach, proposed in [6]. The interval $0 \leqslant \xi \leqslant 1$ is divided into N sections by the points $\xi_j = jh$ (j = 0, 1, ..., N). After the introduction of a difference approximation of the derivatives with an accuracy up to $0(h^2)$, the boundary-value problem (3.1)-(3.3) is reduced to a system of 3N − 3 nonlinear algebraic equations with respect to $p_{1,2}(\xi_j)$ $(j = 1, 2, \ldots, N-1)$, $\theta(\xi_j)$ $(j = 0, 2, 3, \ldots, N-1)$, which is solved by the method of steepest descent. To determine the critical thermal state, we investigate the dependence of the parameter

$$\lambda = \left(\frac{1}{2k} c_2 l^2 \omega_0^{1+\beta} T_2^{\gamma-1}\right)^{1/2} s_0$$

on the maximal temperature along the coordinate $\theta_0 = \theta(0)$, where $\omega_0 = (\pi^2/c_1\rho_0 l^2 T_2^{\gamma})^{\frac{1}{2+\beta}}$ is the first natural frequency of the longitudinal vibrations of an elastic beam. The greatest value on the curve of $\lambda = \lambda(\theta_0)$ is the critical value λ_* of the parameter λ. We find the dependence $\lambda(\theta_0)$ by solution of the above algebraic system, in which we assign the values of θ_0, and λ is regarded as the sought parameter.

The numerical results were obtained for a beam made out of a typical viscoelastic material with the following data [5]: $c_1 = 4.43 \cdot 10^{-14}$ m²/N; $c_2 = 1.56 \cdot 10^{-14}$ m²/N; $\rho = 1214$ kg/m³; $\beta = -0{,}214$; $\gamma = 3{,}21$; $k = 0.15$ W/m·deg; $T_0 = 18.3$°C; $T_1 = -87.2$°C; $l = 0.0762$ m.

Figure 1 gives the dependences $\lambda(\theta_0)$ for $\omega = 1.5 \cdot 10^3$ sec⁻¹ (curve 1) and $\omega = 2.035 \cdot 10^3$ sec⁻¹ (curve 2). We denote the extremal values of λ^0 on the curve $\lambda(\theta_0)$: λ_k^s(k = 1, 2, 3, 4) as maxima; λ_k^i(k = 1, 2, 3) as minima. For curve 1, $\lambda_1^s > \lambda_2^s$; for curve 2, $\lambda_1^s > \lambda_2^s$. The presence of several maxima on the curve $\lambda(\theta_0)$ attests to the existence of several stable (sections of a monotonic rise) and unstable (section of a monotonic decrease) states. These states are determined by the points of intersection of the straight line λ = const with the curve $\lambda(\theta_0)$. The realization of one state or another depends on the initial conditions of the unsteady-state problem. In many situations, there is a possibility of a transition from one state to another. The case where the straight line λ = const lies above the curve $\lambda(\theta_0)$ corresponds to thermal instability. With a zero initial condition, the critical value λ_* of the parameter λ is the greatest of the values of λ_k^s. With a nonzero initial condition, λ_* depends on the position of the point (θ_0, λ); the criterion of thermal instability is the absence of a branch of the dependence $\lambda(\theta_0)$ to the right of this point.

The dependences $\lambda^0(\omega)$ are shown in Fig. 2, where the continuous lines correspond to maxima of the dependences $\lambda(\theta_0)$, and the dashed lines to minima. In the range of frequencies $0 < \omega \leqslant \omega_+$ there exist four stable states; with $\omega_+ < \omega \leqslant \omega_2$, three; with $\omega_2 < \omega \leqslant \omega_4$, two; with $\omega > \omega_4$, one. In the region $\omega_1 < \omega \leqslant \omega_2$ there is a possibility of a transition from the first stable state to the second (with $\lambda > \lambda_1^s$); in the region $\omega_3 < \omega < \omega_4$, from the second stable state to the fourth (with $\lambda > \lambda_2^s$). In the range $0 < \omega \leqslant \omega_1$ $\lambda_* = \lambda_1^s$, for $\omega_1 < \omega \leqslant \omega_3$ $\lambda_* = \lambda_2^s$, for $\omega > \omega_3$ $\lambda_* = \lambda_4^s$. With $\lambda > \lambda_*$, there is thermal instability.

Let us investigate the dependence of the temperature of the beam on the frequency for the subcritical thermal states $\lambda < \lambda_*$. Curves of $\theta_0(\omega)$ are illustrated in Fig. 3 for λ = 0.289 (1) and λ = 0.42 (2), and in Fig. 4 for λ = 0.66, where the continuous lines correspond to stable branches of the curves of $\lambda(\theta_0)$, and the dashed lines to unstable branches. Curves 1 in Fig. 3 is characteristic in that the value of λ = 0.289 lies in the region of change in $\lambda_1^i(\omega)$. With a fixed value of ω, a point of the section 0–I determines the first stable state, and a point of the section III–IV or VI–VII, the second stable state. With a change in ω, there is a jump from point I to point II, as well as from points IV and VII, respectively, to points V and VIII.

For curve 2 in Fig. 3, it is significant that the straight line λ = 0.42 lies above the curve $\lambda_1^i(\omega)$. This leads to a situation in which, with an increase in ω, with satisfaction of the condition $\lambda_1^s < 0.42$, there is a jump from the point IX on the first stable branch to the point X on the second stable branch XI–XII, while, with a decrease in ω, there is no reverse jump from the second stable branch to the first.

The dependence $\theta_0(\omega)$ in Fig. 4 is one of the most interesting cases, where the straight line λ = const intersects all the curves $\lambda^0(\omega)$, with the exception of $\lambda_1^i(\omega)$. For this case, there are two characteristic jumpwise transitions from the lower branches to the upper (from points I and III, respectively, to points II and IV), and two jumps from the upper branches to the lower (from points X and XIII, respectively, to points XI and XIV); if θ_0 is determined by points VIII, IX, or XII, then, with a decrease in ω, there is thermal instability.

The change in the amplitude of the stress in the beam with the frequency is analogous to the change in the temperature.

The results obtained show that the subcritical dynamic behavior of a viscoelastic body, subjected to cyclical loading, is analogous to the behavior of a nonlinear mechanical system with a mild characteristic.

4. The effect of the amplitude dependence of the complex modulus on the thermal behavior of viscoelastic bodies will be studied using the fully established vibrations of a spring-beam with a length l. At its upper end, there is attached a load of mass M, whose displacement we denote by $\tilde{x}_2 = x_2^* e^{i\omega t}$, while the lower end is subjected to a given kinematic perturbation $\tilde{x}_1 = x_1^* e^{i\omega t}$.

The equation of motion for the mass has the form

$$(\varkappa E^* - \omega^2 M)\tilde{\varepsilon} = \omega^2 M \tilde{u}_1, \tag{4.1}$$

where $\tilde{\varepsilon} = (x_2^* - x_1^*)/l$ is the complex deformation; $\tilde{u}_1 = x_1^*/l$; $E^* = E_1^* + iE_2^*$ is the complex Young modulus (here $E_{1,2}^* = E_{1,2}^*(\omega, |\tilde{\varepsilon}|^2, T), |\tilde{\varepsilon}|^2 = \varepsilon_1^2 + \varepsilon_2^2$) $\varkappa$ is a numerical parameter, depending on the form of the beam.

The natural frequency is determined from the equality

$$\varkappa E_1^0 - \omega_0^2 M = 0, \tag{4.2}$$

where $E_1^0 = E_1^*(\omega_0, 0, T_0)$. Analogously $E_2^0 = E_2^*(\omega_0, 0, T_0)$.

We take the expressions for $E_{1,2}^*$ in the form

$$E_{1,2}^* = E_{1,2}^0 f_{1,2}(\omega, T)[1 + \varphi_{1,2}(|\tilde{\varepsilon}|^2)], \tag{4.3}$$

where

$$f_{1,2}(\omega_0, T_0) = 1; \quad \varphi_{1,2}(0) = 0.$$

The equation of the energy balance for the steady-state thermal state under conditions of conductive heat transfer with the surrounding medium at the lateral surface and insulated ends has the form

$$T^* = 1 + \mu f_2(\overline{\omega}, T^*)[1 + \varphi_2(|\tilde{\varepsilon}|^2)]|\tilde{\varepsilon}|^2, \tag{4.4}$$

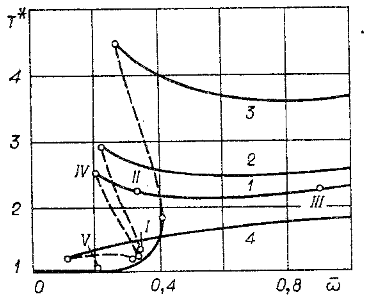

Fig. 5

where

$$T^* = (T - T_1)/(T_0 - T_1);\ \mu = E_2^0 \omega A S/2\alpha_s P(T_0 - T_1);\ \bar{\omega} = \omega/\omega_0;$$

$$T_0 \text{ and } T_1$$

are the initial and reference temperatures; α_s is the heat-transfer coefficient; A is the heat equivalent of mechanical work; P and S are the perimeter and area of the cross section of the beam.

Separating the real and imaginary parts in (4.1), taking account of (4.3), we find

$$\begin{aligned} f_1(\Lambda)[1 + \varphi_1(|\tilde{\varepsilon}|^2)]\varepsilon_1 - \delta f_2(\Lambda)[1 + \varphi_2(|\tilde{\varepsilon}|^2)]\varepsilon_2 - \bar{\omega}^2\varepsilon_1 &= \bar{\omega}^2 u_{11}, \\ f_1(\Lambda)[1 + \varphi_1(|\tilde{\varepsilon}|^2)]\varepsilon_2 + \delta f_2(\Lambda)[1 + \varphi_2(|\tilde{\varepsilon}|^2)]\varepsilon_1 - \bar{\omega}^2\varepsilon_2 &= \bar{\omega}^2 u_{12}, \end{aligned} \quad (4.5)$$

where $\delta = E_2^0(\omega_0, T_0)/E_1^0(\omega_0, T_0)$; $u_{11} = x_{11}/l$; $u_{12} = x_{12}/l$; $(\Lambda) = (\bar{\omega}, T^*)$. Relationships (4.4), (4.5) form a closed system of equations for determination of the functions ϵ_1, ϵ_2, and T*.

With a numerical realization, it was assumed that

$$E_{1,2}^* = \frac{c_{1,2}}{c_1^2 + c_2^2} \omega^{-\beta} (T - T_1)^{-\gamma} \left[1 + \frac{2\alpha_{1,2}}{\pi} \operatorname{arctg} \gamma_{1,2} (\varepsilon_1^2 + \varepsilon_2^2)\right],$$

where $c_1, c_2, \beta, \gamma, \gamma_1, \gamma_2, \alpha_1, \alpha_2$ are constants of the material.

The two-parameter function standing in square brackets gives close to linear ($\alpha_{1,2} > 0$) and mild ($\alpha_{1,2} < 0$) characteristics and makes possible a quantitative description of the amplitude dependence of a broad range of materials.

Figure 5 shows the effect of the dependence of the Young modulus on the amplitude of the deformation on the temperature–frequency characteristics with $\gamma_{1,2} = 10$, $\mu/\bar{\omega} = 100$, $\breve{\mu} = (\mu/\bar{\omega})^{1/2} u_{11} = 4$. The values of c_1, c_2, β, γ, T_0, T_1 are given in Sec. 3. For curves 1-4, the values of $\alpha_{1,2} (\alpha_1 = \alpha_2)$ are equal, respectively, to 0, 1, 10, −1. Curve 1 illustrates the case of an amplitude-independent Young modulus; curves 2, 3 correspond to almost linear characteristics, and curve 4 to a mild characteristic. As was to be expected, the total nonlinearity, being the result of the superposition of mechanical nonlinearity, due to the dependence of the Young modulus, and nonlinearity due to thermomechanical connectedness, is of the mild type.

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